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Recurrent Random Walks and the Absence of Continuous Symmetry Breaking on Graphs

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We consider geometrically disordered systems with a continuous symmetry group G, where the internal degrees of freedom are attached to the vertices of a graph. We show that equilibrium states remain G-invariant at any temperature $T>0$ if a random walk on the graph is recurrent. This generalizes a previous result obtained by Cassi.

KEY WORDS: Recurrent random walks; graphs; Gibbs states; symmetry breaking.

1. INTRODUCTION

The problem of spontaneous symmetry breaking in disordered systems with spatially discrete random structure, such as spin glasses near the percolation threshold, raises technical as well as conceptual questions. Besides the lack of translation invariance, the concept of dimensionality becomes ambiguous, and the notion of a lower critical dimension appears to be lost. Therefore one is forced to look for a more general but possibly less explicit criterion governing symmetry breakdown.

Recently, Cassi[1] considered classical and quantal ferromagnets with internal $O(n)$ symmetry and with spins that reside on the nodes of a generic network. He showed that the spontaneous magnetization in such models vanishes at any finite temperature if a random walk on the underlying network is recurrent, i.e., returns to its starting point with probability one. This result generalizes the familiar dimensionality criterion[2,3] for the absence of long-range order, since random walks on regular d-dimensional lattices are known to be recurrent when $d \leqslant 2$.

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In this paper, which is stimulated by Cassi's work, we extend his result in several respects:

1. The symmetry group is promoted to an arbitrary connected Lie group. In addition, we consider more general two-body interactions which include ferro- as well as antiferromagnetic couplings; moreover, these couplings are not required to be uniformly bounded.

2. We prove that thermal equilibrium states on "recurrent" networks remain invariant under the symmetry operation at any finite temperature. The absence of long-range order is then a special case.

We deal primarily with classical systems, with equilibrium states described by Gibbs measures. The precise statement of our result is formulated in the theorem in Section 2.

The proof of this theorem in Section 3 needs no technical innovations. It combines random walk arguments with an entropy estimation method put forward originally by Araki,[4] Dobrushin and Shlosman,[5] and Pfister.[6] We also adopt ideas from the work of Fröhlich and Pfister[7] on the absence of crystalline order in two dimensions.

Finally, in Section 4, we point out that our theorem (with modifications concerning the notion of equilibrium states) also holds in the case of quantum statistics.

2. DEFINITION OF CLASSICAL MODELS; RESULT

Let $\mathscr{G} = (V, B)$ be a connected graph with a countably infinite set V of vertices or sites and a prescribed subset B of unordered pairs of sites; B is the collection of bonds of $\mathscr{G}$. Thus, by definition, the graph has no multiple bonds. The family of finite subsets of V is denoted by $\mathscr{V}$. For a bond $b = \{x, y\} \in B$, $x, y \in V$, we write $b = xy$, for short. We assume $\mathscr{G}$ to be locally finite, which means that the set of neighbors of every site is finite,

$$\mathrm{Nb}(x) := \{y \in V \backslash \{x\} \mid xy \in B\} \in \mathscr{V}, \qquad x \in V \tag{1}$$

The internal degrees of freedom are introduced by fixing at each site a copy of a compact $\mathscr{C}^2$-manifold M carrying a $\mathscr{C}^2$-operation of a connected Lie group G, $G \times M \to M$, $(g, \alpha) \mapsto g\alpha$. In the example of a classical magnet with n-component spins, M is the sphere S^{n-1}, subject to the natural action of the group $SO(n)$. The total configuration space is given by $M^V = \Omega$. The action L on Ω of the group G is defined in terms of components according to

$$(L_g \omega)_x := (g\omega)_x := g\omega_x, \qquad x \in V, \quad \omega_x \in M \tag{2}$$

The space M^V may be equipped with the *a priori* product measure $\nu_V = \nu^{\otimes V}$, where ν is a G-invariant finite measure on the Borel algebra $\mathscr{B}(M)$ of single-site configurations. In the above example, ν is the area measure on S^{n-1}.

The model interaction is determined by coupling coefficients

$$\kappa\colon B \to \mathbb{R}\backslash\{0\}, \qquad xy \mapsto \kappa_{xy}$$

and by a pair potential $\phi(\alpha, \beta) = \phi(\beta, \alpha)$, $\alpha, \beta \in M$, which is G-invariant, i.e., $\phi(g\alpha, g\beta) = \phi(\alpha, \beta)$ for any $g \in G$, and smooth, i.e., $\phi \in \mathscr{C}^2$.

The boundary of a finite subset $\Lambda \in \mathscr{V}$ is defined by

$$\partial\Lambda := \{x \in \Omega\backslash\Lambda \mid \text{There is a } y \in \Lambda \text{ such that } xy \in B\} \tag{3}$$

Then the local Hamiltonian has the form

$$H_\Lambda(\omega) = \sum_{x,\, y \in \Lambda \cup \partial\Lambda} \kappa_{xy} \phi(\omega_x, \omega_y) \tag{4}$$

A local Hamiltonian is associated with a Boltzmann–Gibbs distribution τ_Λ such that the conditional expectation of an event $A \in \mathscr{B}(\Omega)$ is given by

$$\tau_\Lambda(A \mid \eta) = \frac{1}{Z(\eta)} \int_A \exp[-H_\Lambda(\hat{\zeta\eta})]\, d\nu_\Lambda(\zeta) \tag{5}$$

with the conditional partition function

$$Z(\eta) = \int_{M^\Lambda} \exp[-H_\Lambda(\hat{\zeta\eta})]\, d\nu_\Lambda(\zeta) \tag{6}$$

Here $\hat{\zeta\eta}$ denotes the configuration which coincides with $\zeta \in M^\Lambda$ on Λ and with $\eta \in M^{\Lambda^c}$ on $\Lambda^c = V\backslash\Lambda$. The temperature variable $1/k_\mathrm{B}T$ has been absorbed in H_Λ.

Since our emphasis is on infinite classical systems, equilibrium states will be represented by Gibbs states. We recall that Gibbs states are measures over $\mathscr{B}(\Omega)$ with conditional probabilities on $\Lambda \in \mathscr{V}$ that match the Boltzmann–Gibbs distribution τ_Λ. More precisely, a probability measure $\tau\colon \mathscr{B}(\Omega) \to [0, 1]$ is a Gibbs state corresponding to H_Λ if

$$\tau = r_{\Lambda^c}[\tau] \otimes \tau_\Lambda \tag{7}$$

holds for every $\Lambda \in \mathscr{V}$, where $r_{\Lambda^c}[\tau]$ is the image of τ under the restriction map $r_{\Lambda^c}\colon M^V \to M^{\Lambda^c}$.

Assuming that the collection of Gibbs states of our model is nonempty, we shall prove the following:

Theorem 2.1. If the Markovian random walk on $\mathscr{G}$ with jump probabilities

$$p_{xy} = \begin{cases} |\kappa_{xy}| / \sum_{z \in \mathrm{Nb}(x)} |\kappa_{xz}|, & xy \in B \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

is recurrent, then every Gibbs state specified by the local Hamiltonians H_Λ is G-invariant, i.e., $L_g[\tau] := \tau \circ L_g^{-1} = \tau$, for every $g \in G$.

Before proceeding with the formal proof, let us briefly sketch the basic argument.

Starting with a given Gibbs state τ and some fixed finite set of sites $\Lambda \in \mathscr{V}$, we connect a prescribed probability distribution "far away" from Λ smoothly with a "twisted" distribution over Λ, which is generated by an application of G. The twist is parametrized by a smoothly varying function f on the graph $\mathscr{G}$, which is constructed with the help of local entrance probabilities of a recurrent random walk on $\mathscr{G}$ visiting Λ. This random walk device substitutes for the lack of translation invariance and constitutes a key element of the proof. The theorem is finally established by showing that the relative entropy of the twisted and the original distributions vanishes, which implies that both distributions agree on every $\Lambda \in \mathscr{V}$ and thus on V.

3. PROOF

3.1. Construction of a Twist Map

Let $U = (g_t)_{t \in \mathbb{R}}$ be an arbitrary but fixed subgroup of G, with g_0 the neutral element. To prove the theorem, it suffices to establish that every Gibbs state is U-invariant, since G is connected.

We introduce the twist map $\Gamma_t: \Omega \to \Omega$ by

$$(\Gamma_t(\omega))_x := g_{tf(x)} \omega_x, \qquad \omega \in \Omega, \quad x \in V \tag{9}$$

with the parametrization f supplied by the following lemma.

Lemma 3.1. For each $\varepsilon > 0$ and each nonempty $\Lambda \in \mathscr{V}$ there exists a map $f: V \to [0, 1]$ such that:

(a) $f(x) \neq 0$ for at most a finite number of sites $x \in V$.

(b) $f(x) = 1$ for $x \in \Lambda$.

(c) $\sum_{xy \in B} |\kappa_{xy}| \, [f(x) - f(y)]^2 < \varepsilon$.

Proof of the Lemma.[2] We enumerate the sites of $\mathscr{G}$, $V=\{x_n \mid n\in\mathbb{N}\}$, such that $\Lambda=\{x_1,\ldots,x_i\}$ and $\Delta_k:=\{x_l \mid l\geqslant k\}$. Let P_x denote the probability measure for random walks on $\mathscr{G}$ as in the theorem starting from $x\in\Delta_k^c$. We represent the event "*The random walk enters Λ before Δ_k*" by A_k, $k>i$. Obviously:

(a) $P_x(A_k)=0$ for $x\in\Delta_k$.

(b) $P_x(A_k)=1$ for $x\in\Lambda$.

Hence $f_k(x):=P_x(A_k)$ is a possible candidate for the map f; we have to show that property (c) also holds for f_k with some value of k. For this purpose we note that the events $(A_k)_{k>i}$ form a monotonically increasing sequence, i.e., $A_k\subseteq A_m$ for $i<k\leqslant m$, and the union $\bigcup_{k>i}A_k$ corresponds to the event "*The random walk enters Λ at some time*," since the number of jumps before hitting Λ is finite. Therefore,

$$P_x(A_k)\uparrow P_x\left(\bigcup_{l>i}A_l\right) \qquad \text{as} \quad k\to\infty$$

By assumption, random walks on $\mathscr{G}$ are recurrent, so that $\Lambda\neq\varnothing$ is visited eventually with probability 1. This implies

$$P_x\left(\bigcup_{l>i}A_l\right)=1 \tag{10}$$

and

$$P_x(A_k)\to 1 \qquad \text{as} \quad k\to\infty \tag{11}$$

In order to make the following arguments more transparent without sacrificing rigor, let us imagine $\mathscr{G}$ to be an electric network where each bond is an Ohmic resistor with conductance $|\kappa_{xy}|$; assume the sites $x\in\Lambda$ to be connected with the terminal of a voltage source (voltage $=1$), while the sites $x\in\Delta_k$, $k>i$, are grounded. Denote the electric potential on $\mathscr{G}$ by $(\psi(x)\mid x\in V)$. In the "shell"

$$D:=V\backslash(\Lambda\cup\Delta_k) \qquad (k>i)$$

conservation of charge and Ohm's law lead to

$$\sum_{y\in \mathrm{Nb}(x)} |\kappa_{xy}|\,[\psi(x)-\psi(y)]=0, \qquad x\in D \tag{12}$$

[2] The following arguments are based on the assumption of Theorem 2.1; a more general proof of this lemma may be found in ref. 8.

and therefore

$$\psi(x) = \sum_{y \in \mathrm{Nb}(x)} p_{xy} \psi(y) \tag{13}$$

which means that ψ is a harmonic function on $\mathscr{G}$ with the boundary values

$$\psi(x) = \begin{cases} 1 & \text{for} \quad x \in \Lambda \\ 0 & \text{for} \quad x \in \Delta_k, \quad k > i \end{cases} \tag{14}$$

Let us go back to random walks for a moment. For each path $(x_j)_{j \in \mathbb{N}}$ starting in D, the time independence of the jump probabilities and the Markov property imply

$$(x_j)_{j \in \mathbb{N}} \in A_k \Leftrightarrow (x_{j+1})_{j \in \mathbb{N}} \in A_k \tag{15}$$

which entails

$$P_x(A_k) = \sum_{y \in \mathrm{Nb}(x)} p_{xy} P_y(A_k) \tag{16}$$

Hence $f_k(x) = P_x(A_k)$ is harmonic on $\mathscr{G}$; moreover, it has the same boundary values as $\psi(x)$, so that $f_k(x) = \psi(x)$, as follows from the maximum principle. We now return to the electric network. The total current flowing in the network equals the current passing through the boundary $\partial\Lambda$,

$$\begin{aligned} I_k &= \sum_{x \in \Lambda,\, y \in \partial\Lambda} |\kappa_{xy}| \, [f_k(x) - f_k(y)] \\ &= \sum_{x \in \Lambda,\, y \in \partial\Lambda} |\kappa_{xy}| \, [1 - f_k(y)] \xrightarrow{k \to \infty} 0 \end{aligned} \tag{17}$$

The total power absorbed within the network is $Q_k = \text{voltage drop} \times I_k$. It may also be computed by summing up the power absorbed in each resistor,

$$Q_k = \sum_{xy \in B} |\kappa_{xy}| \, [f_k(x) - f_k(y)]^2 \xrightarrow{k \to \infty} 0 \tag{18}$$

By choosing a sufficiently large value $k > N(\varepsilon)$, we find that f_k does indeed have the properties stated in the lemma.

3.2. Twisted Gibbs State

The map Γ_t gives rise to a twisted Gibbs state

$$\Gamma_t[\tau] = \tau \circ \Gamma_t^{-1}$$

We write $R = \{x \in V \mid f(x) = 0\}$, with f chosen as in the lemma. Then

$$r_R[\tau] = r_R[\Gamma_t[\tau]] \tag{19}$$

where

$$r_R\colon M^V \to M^R, \qquad \omega \mapsto \omega \upharpoonright R$$

is the restriction map. With respect to the measure $\Gamma_t[\tau]$, the conditional probability of an event $A \subseteq M^V$ with indicator function 1_A is expressed by

$$\begin{aligned}
&\Gamma_t[\tau](\omega \in A \mid \omega \upharpoonright R = \eta) \\
&= \frac{1}{Z(\eta)} \int_{M^{R^c}} 1_A(\Gamma_t(\hat{\zeta\eta})) \exp[-H_{R^c}(\hat{\zeta\eta})] \, d\nu_{R^c}(\zeta) \\
&= \frac{1}{Z(\eta)} \int_{M^{R^c}} 1_A(\hat{\zeta\eta}) \exp[-H_{R^c} \circ \Gamma_{-t}(\hat{\zeta\eta})] \, d\nu_{R^c}(\zeta)
\end{aligned} \tag{20}$$

where we used the G-invariance of ν and $r_R \circ \Gamma_t = r_R$. We set

$$W_t = H_{R^c} \circ \Gamma_{-t} - H_{R^c} \tag{21}$$

which reads explicitly

$$W_t(\omega) = \sum_{xy \in B} \kappa_{xy}[\phi(g_{-tf(x)}\omega_x, g_{-tf(y)}\omega_y) - \phi(\omega_x, \omega_y)] \tag{22}$$

Then the unconditional probability of A turns out to be

$$\begin{aligned}
\Gamma_t[\tau](A) &= \int_{M^R} \frac{1}{Z(\eta)} \int_{M^{R^c}} 1_A(\hat{\zeta\eta}) \exp[-W_t(\hat{\zeta\eta})] \\
&\qquad \times \exp[-H_{R^c}(\hat{\zeta\eta})] \, d\nu_{R^c}(\zeta) \, dr_R[\tau](\eta) \\
&= \int_\Omega 1_A e^{-W_t} \, d\tau
\end{aligned} \tag{23}$$

The second equality results from the compatibility relation (7) for Gibbs states and shows that the image measure $\Gamma_t[\tau]$ has the density

$$\frac{d\Gamma_t[\tau]}{d\tau} = e^{-W_t} \tag{24}$$

relative to τ.

3.3. Energy Estimate

For estimating the entropy, we need a bound on the excess energy created by twisting $|W_t|$ for fixed values of t. We expect that $|W_t| = O(t)$ due to "pre-twisted" configurations. It is convenient to eliminate a possible linear term in $|W_t|$ and to consider $|W_t + W_{-t}|$ instead.[7] For this purpose we write

$$\phi(g_t\alpha, \beta) =: F(t, \alpha, \beta), \qquad \alpha, \beta \in M \tag{25}$$

The Taylor formula yields

$$\begin{aligned} &F(t, \alpha, \beta) + F(-t, \alpha, \beta) - 2F(0, \alpha, \beta) \\ &\quad = t^2 \int_{-1}^{1} (1 - |a|)\, \partial_1^2 F(at, \alpha, \beta)\, da =: K_t(\alpha, \beta) \end{aligned} \tag{26}$$

Here, $\partial_1 F$ denotes the partial derivative of F with respect to its first argument. Since we assume M to be compact and $\partial_1^2 F$ to be continuous,

$$|\partial_1^2 F(t, \alpha, \beta)| = |\partial_1^2 F(0, g_t\alpha, \beta)| \leqslant \sup_{\alpha, \beta \in M} \partial_1^2 F(0, \alpha, \beta)| =: C \tag{27}$$

with $C < \infty$. Consequently, $|K_t(\alpha, \beta)| \leqslant Ct^2$, and

$$|\phi(g_t\alpha, \beta) + \phi(g_{-t}\alpha, \beta) - 2\phi(\alpha, \beta)| \leqslant Ct^2 \tag{28}$$

for all $\alpha, \beta \in M$, $t \in \mathbb{R}$ with $C > 0$. This inequality provides a bound on $|W_t + W_{-t}|$, required below.

Remark. The proof of Theorem 2.1 may be completed already at this point by combining the energy estimate (28) with an argument given, for instance, in ref. 11. However, we prefer not to follow this technically elegant but less convenient shortcut.

3.4. Entropy Estimate

Let us first recall some familiar facts concerning relative entropies. Consider two probability measures μ_1 and μ_2 on a common space $(\Omega, \mathscr{A})$, where $\mathscr{A}$ is a σ-algebra of events, and $\mu_1(\Omega) = \mu_2(\Omega) = 1$. We assume μ_1 to be absolutely continuous with respect to μ_2, with the density $d\mu_1/d\mu_2$. The entropy of μ_1 relative to μ_2 is defined by

$$S(\mu_1 \mid \mu_2) := -\int \ln\left(\frac{d\mu_1}{d\mu_2}\right) d\mu_2 \tag{29}$$

and has the following properties:

(E) $S(\mu_1 \mid \mu_2) \leqslant 0$, with equality if and only if $\mu_1 = \mu_2$.

Let $\mathscr{A}' \subseteq \mathscr{A}$ be a sub-σ-algebra, with $\mu'_1 = \mu_1 \upharpoonright \mathscr{A}'$ and $\mu'_2 = \mu_2 \upharpoonright \mathscr{A}'$ denoting the restrictions of μ_1 and μ_2 on $\mathscr{A}'$. Then:

(M) $S(\mu'_1 \mid \mu'_2) \geqslant S(\mu_1 \mid \mu_2)$.

As a final step in proving the theorem, we estimate the relative entropy of a Gibbs state τ and its image $\Gamma_t[\tau]$. In order to apply Lemma 3.1 and the bound (28), we consider

$$\begin{aligned} &S(\Gamma_t[\tau] \otimes \Gamma_{-t}[\tau] \mid \tau \otimes \tau) \\ &\quad = S(\Gamma_t[\tau] \mid \tau) + S(\Gamma_{-t}[\tau] \mid \tau) \\ &\quad = \int_\Omega W_t \, d\Gamma_t[\tau] + \int_\Omega W_{-t} \, d\Gamma_{-t}[\tau] \\ &\quad = \int_\Omega W_t \circ \Gamma_t \, d\tau + \int_\Omega W_{-t} \circ \Gamma_{-t} \, d\tau \\ &\quad = \int_\Omega (2H_{R^c} - H_{R^c} \circ \Gamma_t - H_{R^c} \circ \Gamma_{-t}) \, d\tau \end{aligned} \tag{30}$$

We note that the integrals are well-defined, since $R^c \in \mathscr{V}$. Using the inequality (28) in combination with the lemma, we estimate

$$\begin{aligned} &S(\Gamma_t[\tau] \otimes \Gamma_{-t}[\tau] \mid \tau \otimes \tau) \\ &\quad = \int_\Omega \sum_{xy \in B} \kappa_{xy} \\ &\qquad \times [2\phi(\omega_x, \omega_y) - \phi(g_{tf(x)}\omega_x, g_{tf(y)}\omega_y) - \phi(g_{-tf(x)}\omega_x, g_{-tf(y)}\omega_y)] \, d\tau \\ &\quad \geqslant -\int_\Omega \sum_{xy \in B} |\kappa_{xy}| \\ &\qquad \times |\phi(g_{t(f(x)-f(y))}\omega_x, \omega_y) + \phi(g_{t(f(y)-f(x))}\omega_x, \omega_y) - 2\phi(\omega_x, \omega_y)| \, d\tau \\ &\quad \geqslant -Ct^2 \int_\Omega \sum_{xy \in B} |\kappa_{xy}| \, [f(x) - f(y)]^2 \, d\tau \\ &\quad = -Ct^2 \int_\Omega \varepsilon \, d\tau = -Ct^2 \varepsilon \end{aligned} \tag{31}$$

On $\Lambda = \{x \in V \mid f(x) = 1\}$ the uniformly twisted Gibbs state $L_{g_t}[\tau]$ agrees with $\Gamma_t[\tau]$,

$$(r_\Lambda \circ L_{g_t})[\tau] = (r_\Lambda \circ \Gamma_t)[\tau] \tag{32}$$

For the probability measures

$$\mu_1 := (r_\Lambda \circ L_{g_t})[\tau] \otimes (r_\Lambda \circ L_{g_{-t}})[\tau] \tag{33}$$

$$\mu_2 := r_\Lambda[\tau] \otimes r_\Lambda[\tau] \tag{34}$$

the monotonicity (M) of the relative entropy yields

$$S(\mu_1 \mid \mu_2) \geqslant S(\Gamma_t[\tau] \otimes \Gamma_{-t}[\tau] \mid \tau \otimes \tau) \geqslant -Ct^2\varepsilon \tag{35}$$

Since $L_{g_t}[\tau]$ is independent of the choice of $\varepsilon > 0$, we infer that $S(\mu_1 \mid \mu_2) \geqslant 0$; but the extremum property (E) requires $S(\mu_1 \mid \mu_2) \leqslant 0$; therefore $\mu_1 = \mu_2$, and thus

$$\{r_\Lambda \circ L_{g_t})[\tau] = r_\Lambda[\tau] \tag{36}$$

holds for every $\Lambda \in \mathscr{V}$. The algebra of measurable events $\mathscr{B}(M)^{\otimes V}$ is generated by

$$\mathscr{E} := \{r_\Lambda^{-1}[A] \mid \Lambda \in \mathscr{V}, A \in \mathscr{B}(M)^{\otimes \Lambda}\} \tag{37}$$

which is closed under intersection. By the uniqueness theorem for the extension of measures,[9] the equality

$$L_{g_t}[\tau] \restriction \mathscr{E} = \tau \restriction \mathscr{E}$$

implies

$$L_{g_t}[\tau] = \tau$$

whereby the proof is completed.

4. QUANTAL SYSTEMS

Some years ago, Bonato *et al.*[10] demonstrated that the invariance of equilibrium states of two-dimensional classical and quantum models with continuous internal symmetry on a regular lattice follows from Bogoliubov's inequality without requiring translational invariance of the Hamiltonian. Here, we point out that the lattice can be replaced by a generic graph so that the quantal analog of our theorem is obtained without further efforts. To outline the argument, we use the terminology of ref. 10.

Equilibrium states of quantal systems on a countably infinite graph $\mathscr{G}$ may be represented by expectation value functionals on the (norm closed) C^*-algebra of observables $\mathscr{A}$, $\omega: \mathscr{A} \to \mathbb{R}$,

$$\mathscr{A} = \overline{\bigcup_{\Lambda \in \mathscr{V}} \mathscr{A}_\Lambda}$$

where $\mathscr{A}_\Lambda$ is the C^*-algebra of observables on $\Lambda \in \mathscr{V}$. The continuous symmetry is described by a one-parameter group of automorphisms $(\sigma_s)_{s \in \mathbb{R}}$ such that $\sigma_s \mathscr{A} = \mathscr{A}$ where $\sigma_s := \bigotimes_{x \in V} \sigma_s(x)$. In this frame a state ω is said to be symmetric if

$$\omega(\sigma_s A) = \omega(A)$$

for all $A \in \mathscr{A}$, or equivalently,

$$\frac{d}{ds}(\sigma_s A)\bigg|_{s=0} = 0$$

for all $A \in \bigcup_{\Lambda \in \mathscr{V}} \mathscr{A}_\Lambda$.

The Hamiltonian is introduced by the *formal* expression

$$H = \sum_{\Lambda \subset V} H(\Lambda) \tag{38}$$

and we adopt the assumption on $H(\Lambda)$ made by Bonato *et al.*[10] In particular, $\sigma_s H(\Lambda) = H(\Lambda)$, $\Lambda \in \mathscr{V}$. It follows that the terms

$$j(x, y) := \frac{d}{ds}\frac{d}{dt}\,\omega(\sigma_s(x)\,\sigma_t(y)\,H)\bigg|_{s=0,\,t=0} \tag{39}$$

which may be interpreted as coupling coefficients, are well defined and satisfy:

(i) $j(x, y) = j(y, x)$, $x, y \in V$.

(ii) $\sum_{y \in V} j(x, y) = 0$, $x \in V$.

However, we restrict the local Hamiltonians further by requiring:

(iii) $j(x, y) = 0$ if $xy \notin B$.

By employing the couplings $(j_{xy})_{xy \in B}$, we define a random walk as in the theorem. If this random walk is recurrent, then

$$\sum_{xy \in B} |j_{xy}| \cdot |f(x) - f(y)|^2 < \varepsilon \tag{40}$$

with f as specified in the lemma, so that we may define a twist map,

$$\sigma_s(f) := \bigotimes_{x \in V} \sigma_{sf(x)}(x): \mathscr{A} \to \mathscr{A} \tag{41}$$

As a basic assumption,[10] the states ω satisfy Bogoliubov's inequality, which yields, in particular,

$$\left| \frac{d}{ds} \omega(\sigma_s(f) A) \right|^2 \Bigg|_{s=0} \leqslant \beta\omega\left(\frac{1}{2}(AA^* + A^*A)\right) \omega(K) \tag{42}$$

where $\beta = 1/k_B T$,

$$K = \frac{d}{ds}\frac{d}{dt}(\sigma_s(f))\,\sigma_t(f) H)\Bigg|_{s=0,\, t=0}$$

and

$$\omega(K) = \sum_{x, y \in V} f(x) f(y)\, j(x, y)$$

From (i), (ii), and inequality (40)

$$|\omega(K)|^2 \leqslant \tfrac{1}{2} \sum_{x, y \in V} |j(x, y)|\, [f(x) - f(y)]^2 < \tfrac{1}{2}\varepsilon \tag{43}$$

Since with $A \in \mathscr{A}_\Lambda$ and $f(x) = 1$ at every site $x \in \Lambda$ we have

$$\frac{d}{ds}\omega(\sigma_s(f) A)\Bigg|_{s=0} = \frac{d}{ds}\omega(\sigma_s A)\Bigg|_{s=0} \tag{44}$$

for every $\Lambda \in \mathscr{V}$, we infer from the inequalities (42) and (43) that

$$\frac{d}{ds}\omega(\sigma_s A)\Bigg|_{s=0} = 0, \qquad A \in \mathscr{A}_\Lambda \tag{45}$$

holds for every $\Lambda \in \mathscr{V}$ and thus on $\mathscr{A}$, which implies the invariance of the equilibrium states.

Concluding, we note that the algebraic approach of this section based on Bogoliubov's inequality also applies to classical systems.

5. CONCLUDING REMARK

The recurrence criterion for the absence of continuous symmetry breaking is clearly more general but also less practical than the notion of a lower critical dimension. The reason is that proving the recurrence of a random walk on a given graph may not be easy. In some cases this task can be simplified by a suitable modification of the couplings determining the jump probabilities. Consider, for instance, a second set of couplings $(\bar{\kappa}_{xy})_{xy \in B}$ which provides upper bounds for the original ones,

$$|\kappa_{xy}| \leqslant |\bar{\kappa}_{xy}|, \qquad xy \in B \tag{46}$$

If the random walk on $\mathscr{G}$ with jump probabilities

$$\bar{p}_{xy} = \begin{cases} |\bar{\kappa}_{xy}| / \sum_{z \in \mathrm{Nb}(x)} |\bar{\kappa}_{xz}|, & xy \in B \\ 0, & \text{otherwise} \end{cases} \tag{47}$$

is recurrent, then the lemma also holds with the original couplings, since

$$\sum_{xy \in B} |\kappa_{xy}| \cdot |f(x) - f(y)|^2 \leqslant \sum_{xy \in B} |\bar{\kappa}_{xy}| \cdot |f(x) - f(y)|^2 < \varepsilon \tag{48}$$

and the G-invariance of the equilibrium states with the couplings $(\kappa_{xy})_{xy \in B}$ follows.

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